Permutable subgroups of groups

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Naples, 8th October, 2015
If $A, B \leq G$, $A$ permutes with $B$ when $AB = BA$, that is, $AB$ is a subgroup of $G$.

$A$ is permutable or quasinormal in $G$ if $H$ permutes with all subgroups of $G$ (Ore, 1939).
Lemma

Let $\mathcal{X}$ be a family of subgroups of a group $G$. If a subgroup $A$ of $G$ permutes with all subgroups $X \in \mathcal{X}$, then it also permutes with their join $\langle \mathcal{X} \rangle$.

Therefore a subgroup $A$ of a periodic group $G$ is permutable in $G$ if and only if $A$ permutes with every $p$-subgroup of $G$ for all $p \in \pi(G)$. 
Let $\pi$ a set of primes. A subgroup $A$ of a periodic group $G$ is called:

**Definition**

- $\pi$-**permutable** in $G$ if $A$ permutes with every $q$-subgroup of $G$ for all $q \in \pi$.
- $\pi$-**S-permutable** or $\pi$-**S-quasinormal** in $G$ if $A$ permutes with every Sylow $q$-subgroup of $G$ for all $q \in \pi$ (Kegel 1962).
If $\pi$ is the set of all primes, then the $\pi$-permutable subgroups are just the permutable subgroups; the $\pi$-$S$-permutable subgroups are called $S$-permutable. In the case when $\pi = \pi(G) \setminus \pi(A)$, then $A$ is called semipermutable (respectively, $S$-semipermutable) in $G$ (Chen 1987).
Theorem (Kegel (1962), Deskins (1963))

If $A$ is an $S$-permutable subgroup of a finite group $G$, then $A/A_G$ is contained in the Fitting subgroup of $G/A_G$. In particular, $A$ is subnormal in $G$.

In particular, every permutable subgroup is subnormal (Ore, 1939).
Theorem (Maier and Schmid, 1973)

If $A$ is a permutably subgroup of a finite group $G$, then $A/A_G$ is contained in the hypercentre of $G/A_G$.

- There are examples of permutably subgroups $A$ of finite groups $G$ such that $A/A_G$ is not abelian (Thompson, 1967).
- There is no bound for the nilpotency class of a core-free permutably subgroup (Bradway, Gross and Scott, 1971).
- There is no bound for the derived length of a core-free permutably subgroup (Stonehewer, 1974).
A subgroup $A$ of a group $G$ is called hypercentrally embedded in $G$ if $A/A_G$ is contained in the hypercentre of $G/A_G$.

Every S-permutable subgroup of a finite group is hypercentrally embedded in $G$, but the converse is not true in general.
Theorem (Schmid, 1998)

Let $A$ be an $S$-permutable subgroup of a finite soluble group $G$. Then $A$ hypercentrally embedded in $G$ if and only if $A$ permutes with some system normaliser of $G$. 
Theorem (Stonehewer, 1972)

If $A$ is a permutable subgroup of a group $G$, then $A$ is ascendant in $G$. If $G$ is finitely generated, then $A$ is subnormal in $G$.

Kargapolov (1961) showed that S-permutable subgroups of locally finite groups do not have to be ascendant.
Theorem

Let $A$ be an $S$-permutable subgroup of a periodic group. Then $A$ is ascendant if:

- $G$ is locally finite with $\min_p$ for all $p$ (Robinson, Ischia 2010).
- $G$ is hyperfinite (B-B, Kurdachenko, Otal and Pedraza, 2010).
Maier and Schmid’s theorem does not hold in the general case (Busetto and Napolitani, 1992). For locally finite groups, we have:

**Theorem (Celentani, Leone and Robinson, 2006)**

*If $A$ is a permutably subgroup of a locally finite group $G$, then $A/A_G$ is locally nilpotent, and their Sylow subgroups are finite provided that $G$ has min-$p$ for all primes $p$.***
Theorem (Celentani, Leone and Robinson, 2006)

Let $A$ be a permutable subgroup of a locally finite Kurdachenko group. Then $A^G/A_G$ is finite and it is contained in a term of the upper central series of $G/A_G$ of finite ordinal type.
Permutability
S-permutable embeddings

Definitions (Asaad and Heliel, 2003)

- We say that $\mathcal{Z}$ is a complete set of Sylow subgroups of a periodic group $G$ if for each prime $p \in \pi(G)$, $G$ contains exactly a Sylow $p$-subgroup $G_p$ of $G$.

- If $\mathcal{Z}$ is a complete set of Sylow subgroups of a group $G$, we say that a subgroup $A$ of a group $G$ is $\mathcal{Z}$-permutable if $A$ permutes with all subgroups in $\mathcal{Z}$. 

If $G$ is a periodic group, a complete set of Sylow subgroups of $G$ composed of pairwise permutable subgroups $G_p$, $p$ prime, is called a **Sylow basis**. The existence of Sylow basis characterises solubility in the finite universe.
Theorem (Almestady, B-B, Esteban-Romero and Heliel, 2015)

Let $A$ be a subnormal $3$-permutable subgroup of a finite group $G$. Then $A/A_G$ is soluble. If $3$ is a Sylow basis, then $A/A_G$ is nilpotent.
Theorem (B-B, Camp-Mora and Kurdachenko, 2014)

Let $A$ be an $S$-permutable subgroup of a locally finite group $G$. If $A$ is ascendant in $G$, then $A^G/A_G$ is locally nilpotent.
Lemma

Let $H$ and $S$ be periodic subgroups of a group $G$. Suppose that $H$ is an ascendant subgroup of $G$ permuting with $S$. If $\pi$ is a set of primes containing $\pi(S)$, then $O^\pi(H) = O^\pi(HS)$. 
Theorem (see Doerk and Hawkes (1992), I, 4.29)

If \( \mathfrak{S} \) is a Sylow basis of the finite soluble group \( G \), then the set of all \( \mathfrak{S} \)-permutable subgroups of \( G \) is a lattice.
Theorem (see Doerk and Hawkes (1992), I, 4.29)

*If $\mathfrak{Z}$ is a Sylow basis of the finite soluble group $G$, then the set of all $\mathfrak{Z}$-permutable subgroups of $G$ is a lattice.*
Theorem (Almestady, B-B, Esteban-Romero and Heliel, 2015)

If $\mathcal{P}$ is a complete set of Sylow subgroups of a finite group $G$, then the set of all subnormal $\mathcal{P}$-permutable subgroups of $G$ is a sublattice of the lattice of all subgroups of $G$. 
Let $p$ be a prime and $U$ and $V$ subgroups of a finite group $G$. If $U$ and $V$ permute with a Sylow $p$-subgroup $G_p$ of $G$ and $U$ is subnormal in $G$, then $U \cap V$ permutes with $G_p$. 
Corollary (Kegel, 1962)

*S-permutable subgroups of a finite group G form a sublattice of the subgroup lattice of G.*

According to a result of Dixon, Kegel’s lattice result also holds for radical locally finite groups with min-$p$ for all $p$.

We do not know whether

- $3$-permutable subgroups is a sublattice of the subgroup lattice of a finite $G$, even in finite soluble groups.
- Kegel’s result holds for locally finite groups.
Definition

A group $G$ is called a $T_0$-group if the Frattini factor group $G/\Phi(G)$ is a $T$-group, that is, a group in which normality is a transitive relation.
Theorem (B-B, Beidleman, Esteban-Romero and Ragland, 2014)

Let $G$ be a group with nilpotent residual $L$, $\pi = \pi(L)$. Let $\theta_1$ (respectively $\theta_2$) denote the set of all primes $p$ in $\theta = \pi'$ such that $G$ has a non-cyclic (respectively cyclic) Sylow $p$-subgroup. Then every maximal subgroup of $G$ is S-semipermutable if and only if $G$ satisfies the following:

1. $G$ is a $T_0$-group.
2. $L$ is a nilpotent Hall subgroup of $G$.
3. If $p \in \pi$ and $P$ is a Sylow $p$-subgroup of $G$, then a maximal subgroup of $P$ is normal in $G$. 
5. Let $p$ and $q$ be distinct primes with $p \in \theta_1$ and $q \in \theta$. Further, let $P$ be a Sylow $p$-subgroup of $G$ and $Q$ a Sylow $q$-subgroup of $G$. Then $[P, Q] = 1$.

6. Let $p$ and $q$ be distinct primes with $p \in \theta_2$ and $q \in \theta$. Further, let $P$ be a Sylow $p$-subgroup of $G$, $Q$ a Sylow $q$-subgroup of $G$ and $M$ the maximal subgroup of $P$. Then $QM = MQ$ is a nilpotent subgroup of $G$. 

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Definition

Let $p$ be a prime. A group $G$ is $p$-supersoluble if there exists a series of normal subgroups of $G$

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

such that the factor group $G_i/G_{i-1}$ is cyclic of order $p$ or a $p'$-group for $1 \leq i \leq r$.

A group $G$ is supersoluble if and only if $G$ is $p$-supersoluble for all primes $p$. 
Theorem (Berkovich and Isaacs, 2014)

Let $p$ be a prime and $e \geq 3$. Assume that a Sylow $p$-subgroup of a group $G$ is non-cyclic with order exceeding $p^e$. If every non-cyclic subgroup of $G$ of order $p^e$ is $S$-semipermutable in $G$, then $G$ is $p$-supersoluble.
Theorem (Isaacs, 2014)

Let $\pi$ be a set of primes. The normal closure $A^G$ of an $S$-semipermutable $\pi$-subgroup $A$ of a group $G$ contains a nilpotent $\pi$-complement, and all $\pi$-complements are conjugate. Also, if $\pi$ consists of a single prime, $A^G$ is soluble. As a consequence, if $A$ is a nilpotent Hall subgroup of $G$ and $A$ is $S$-semipermutable, then $A^G$ is soluble.
Theorem (B-B, Li, Su and Xie, 2014)

Let $\pi$ and $\rho$ be sets of primes. If $A$ be a $\pi$-$S$-permutable $\rho$-subgroup of a group $G$, then $A^G/O_\rho(A^G)$ has nilpotent Hall $\pi$-subgroups.
Lemma (Wielandt)

Let $G$ be a group and let $A$ and $B$ be subgroups of $G$ such that $AB^g = B^g A$ for all $g \in G$. Then $[A, B]$ is subnormal in $G$. 
Permutability

\( \pi \)-permutability

Theorem (B-B, Li, Su and Xie, 2014)

*If A is a nilpotent \( \pi \)-S-permutable subgroup of a group G, then \( O^{\pi'}(A^G) \) is soluble.*
Corollary

If $A$ is a nilpotent $\rho$-subgroup of a group $G$ and $A$ is $S$-semipermutable in $G$, then $O^\rho(A^G)$ is soluble. In particular, the normal closure of every nilpotent $S$-semipermutable subgroup of $G$ of odd order is soluble.
Theorem (Isaacs, 2014)

Let $A$ be a subgroup of odd order of a finite group $G$. If $A$ permutes with every $2$-subgroup of $G$, then $A^G$ is of odd order.
Theorem (B-B, Li, Su and Xie, 2014)

Suppose that $p$ is a prime such that $(p - 1, |G|) = 1$. Let $A$ be a $p'$-subgroup of a finite group $G$, and assume that $A$ is permutable with every $p$-subgroup of $G$. Then $A^G$ is a $p'$-group.
Theorem (B-B, Li, Su and Xie, 2014)

Let $p$ be a prime and let $A$ be a $p'$-subgroup of a finite group $G$. Assume that $A$ permutes with every $p$-subgroup of $G$. Then every chief factor of $A^G$ whose order is divisible by $p$ is simple and isomorphic to one of the following groups:

1. $C_p$,
2. $A_p$,
3. $\text{PSL}(n, q)$, $n > 2$ prime, $p = \frac{q^n-1}{q-1}$, or $\text{PSL}(2, 11)$, $p = 11$,
4. $M_{23}$, $p = 23$, or $M_{11}$, $p = 11$.

If, moreover, $A^G$ is $p$-soluble, then all $p$-chief factors are $A^G$-isomorphic when regarded as $A^G$-modules by conjugation.
Corollary (B-B, Li, Su and Xie, 2014)

Let \( p \) be a prime and let \( A \) be a \( p' \)-subgroup of a finite \( G \). Assume that \( A \) is permutable with every \( p \)-subgroup of \( G \). If \( A^G \) is \( p \)-soluble, then \( A^G / O_{p'}(A^G) \) is a soluble PST-group and either \( A^G / O_{p'}(A^G) \) is nilpotent or the Sylow \( p \)-subgroups of \( A^G \) are abelian.